Matrični metod

$X' = AX$

$X = \Phi(t) \cdot C$

$x' = ax$

$\frac{dx}{x} = a\,dt$

$x = ce^{at}$

$\boxed{X = e^{At} \cdot C}$ - opšti oblik

$e^{t} = 1 + t + \frac{t^2}{2!} + \dots + \frac{t^n}{n!} + \dots$

$e^{At} = E + At + \frac{t^2}{2!} A^2 + \dots + \frac{t^n}{n!} A^n + \dots$

1. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$

$$\Rightarrow \quad A^{2k} = E \quad , \quad A^{2k+1} = A \quad , \quad k \in \mathbb{N}$$

$$e^{At} = E + tA + \frac{t^2}{2!} E + \frac{t^3}{3!} A + \dots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots =$$

$$= \begin{pmatrix} 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots & t + \frac{t^3}{3!} + \dots \\ t + \frac{t^3}{3!} + \dots & 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \end{pmatrix} = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix}$$

2. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$A^2 = -E$$

$$A^3 = -A$$

$$A^4 = E$$

$$e^{At} = E + At + \frac{t^2}{2!} E - \frac{t^3}{3!} A + \frac{t^4}{4!} E + \dots$$

$$= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots & t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots \\ -t + \frac{t^3}{3!} - \frac{t^5}{5!} \dots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$x = e^{At} \cdot c$$

$$\Phi'(t) = \Phi(t) \cdot c$$

$$(e^{At})' = \left(E + At + \frac{t^2}{2!} A^2 + \dots + \frac{t^n}{n!} A^n + \dots\right)' =$$

$$= A + tA^2 + \frac{t^2}{2!} A^3 + \dots + \frac{t^{n-1}}{(n-1)!} A^n + \dots =$$

$$= A\left(E + tA + \frac{t^2}{2!} A^2 + \dots + \frac{t^{n-1}}{(n-1)!} A^{n-1} + \dots\right) = A \cdot e^{At}$$

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$$\Rightarrow \boxed{\Phi(t) = e^{At}}$$

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix} \quad \text{- Jordanova matrica}$$

$$x = Ty \quad \Rightarrow \quad x' = Ty'$$

matrica od vlastitih vektora i njihovih pridruženih vektora

$$x' = Ax$$

$$T^{-1} /\ Ty' = ATy$$

$$y' = \underbrace{T^{-1}AT}_{J}\, y$$

$$J = \begin{pmatrix} J(\lambda_1) & & & 0 \\ & J(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J(\lambda_r) \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}$$

$$\lambda, r: \qquad J(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{pmatrix}$$

$$y' = Jy, \quad y = e^{\lambda t}$$

$$\Rightarrow J^{\omega}(\lambda) = \begin{pmatrix} \lambda^{\omega} & \binom{\omega}{1}\lambda^{\omega-1} & \cdots & \binom{\omega}{r}\lambda^{\omega-(r-1)} \\ & \ddots & & \vdots \\ & & & \binom{\omega}{1}\lambda^{\omega-1} \\ 0 & & & \lambda^{\omega} \end{pmatrix}$$

$$e^{J(\lambda)t} = E + tJ + \frac{t^2}{2!}J^2 + \ldots = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{r-1}}{(r-1)!}e^{\lambda t} \\ & \ddots & & \vdots \\ & & & te^{\lambda t} \\ 0 & & & e^{\lambda t} \end{pmatrix} =$$

$$= e^{\lambda t}\begin{pmatrix} 1 & t & \cdots & \frac{t^{r-1}}{(r-1)!} \\ & \ddots & & \vdots \\ & & & t \\ 0 & & & 1 \end{pmatrix}$$

$$e^{Jt} = \begin{pmatrix} e^{J(\lambda_1)} & & & \\ & e^{J(\lambda_2)} & & 0 \\ 0 & & \ddots & \\ & & & e^{J(\lambda_r)} \end{pmatrix}$$

$J = T^{-1}AT$

$e^{JT} = T^{-1}e^{At}T$

$e^{At} = Te^{Jt}T^{-1}$

1. $X' = AX$, $A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$

$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = 0$

$\lambda_1 = 1 \longrightarrow h_1 : Ah_1 = h_1$

$\lambda_2 = 5$

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} h_{11} \\ h_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$h_{11} = -h_{21}$

$h_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 5 \longrightarrow h_2 : Ah_2 = 5h_2$

$$\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} h_{12} \\ h_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$3h_{12} = h_{22}$

$h_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$J = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

$J = T^{-1}AT$, $T = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$e^{Jt} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{5t} \end{pmatrix}$

$e^{At} = T e^{Jt} T^{-1}$

rješenje: $x = e^{At} c$

$x = Ty$

$x' = Ax$

$Ty' = ATy$

$y' = Jy$

$y = e^{Jt} c$

2 $x' = Ax$, $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$

$\det(A - \lambda E) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = 0$

$\lambda_1 = \lambda_2 = 3$

$\text{rang}(A - 3E) = \text{rang}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = 1$

$m = 2 - 1 = 1 < 2$ - možemo naći samo 1 vl. vektor.

Vector

$J = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

h_1: $Ah_1 = 3h_1$, $h_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

h_2: $Ah_2 = 3h_2 + h_1$

$(A - 3E)h_2 = h_1$, $h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$x = Ty$

$x' = Ax$

$Ty' = ATy \Rightarrow y' = \underbrace{T^{-1}AT}_{J}y$ ... nađemo T, množimo A

$y = e^{Jt} c$

$$e^{Jt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e^{3t}$$

$x = T e^{Jt} c$

3. $x' = Ax$, $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

4. $x' = Ax$, $A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$

5. $x' = Ax$, $A = \begin{pmatrix} -1 & 1 & -2 \\ 4 & 1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$

$p(\lambda) = \det(A - \lambda E) = 0$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = -1$

h_1: $Ah_1 = h_1 \Rightarrow h_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

h_2: $Ah_2 = 2h_2 \Rightarrow h_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

h_3: $Ah_3 = -h_3 \Rightarrow h_3 = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$

$$T = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{pmatrix}$$

$X = TY$

$X' = TY'$

$X' = AX$

$T^{-1} / \; TY' = ATY$

$Y' = \underbrace{T^{-1}AT}_{J} Y$

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$Y = e^{Jt} C \quad , \quad e^{Jt} = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{pmatrix} \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$X = \begin{pmatrix} e^{t} & e^{2t} & -e^{-t} \\ e^{t} & 0 & 3e^{-t} \\ e^{t} & e^{2t} & 5e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

4. $$A = \begin{pmatrix} 4 & -1 & -1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$\det(A - \lambda E) = 0$

$\lambda_1 = 2 \quad , \quad \lambda_2 = 3 \quad , \quad r_2 = 2$

$(A - \lambda_1 E) h_1 = 0$

$$\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$h_{11} = h_{13}$

$h_{11} = h_{12}$ $\Rightarrow$ $h_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$, $r = 2$

$$\mathrm{rang}\,(A-3E) = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = 1$$

$m = 3 - 1 = 2$

$(A-3E)h_2 = 0$

$$\begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} h_{21} \\ h_{22} \\ h_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow h_{21} = h_{22} + h_{23}$$

za $h_{22} = 0$, $h_{23} = 1 \Rightarrow h_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

za $h_{22} = 1$, $h_{23} = 0 \Rightarrow h_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$X = TY$

$X' = TY'$, $X' = AX$

$TY' = ATY$

$Y' = \underbrace{T^{-1}AT}_{J}Y$

$$J = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$Y = e^{Jt}C = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} C_1e^{2t} \\ C_2e^{3t} \\ C_3e^{3t} \end{pmatrix}$$

$$X = TY = \begin{pmatrix} C_1e^{2t} + C_2e^{3t} + C_3e^{3t} \\ C_1e^{2t} + C_3e^{3t} \\ C_1e^{2t} + C_2e^{3t} \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} e^{2t} & e^{3t} & e^{3t} \\ e^{2t} & 0 & e^{3t} \\ e^{2t} & e^{3t} & 0 \end{pmatrix}$$